Stochastic space-time and quantum theory

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Much of quantum mechanics may be derived if one adopts a very strong form of Mach's principle such that in the absence of mass, space-time becomes not flat but stochastic. This is manifested in the metric tensor which is considered to be a collection of stochastic variables. The stochastic-metric assumption is sufficient to generate the spread of the wave packet in empty space. If one further notes that all observations of dynamical variables in the laboratory frame are contravariant components of tensors, and if one assumes that a Lagrangian can be constructed, then one can obtain an explanation of conjugate variables and also a derivation of the uncertainty principle. Finally, the superposition of stochastic metrics and the identification of $\sqrt{-g}$ in the four-dimensional invariant volume element $\sqrt{-g}\,dV$ as the indicator of relative probability yields the phenomenon of interference as will be described for the two-slit experiment.

I. INTRODUCTION

When considering the quantum and relativity theories, it is clear that only one of them, namely relativity, can be considered, in the strict sense, a theory. Quantum mechanics, eminently successful as it is, was laboriously developed by a number of people as an operational description of physical phenomena. It is composed of several principles, equations, and a set of interpretive postulates.[1] These elements of quantum mechanics are justifiable only in that they work. Attempts[2,3] to create a complete, self-consistent theory for quantum mechanics are largely unconvincing. There are in addition a number of points where quantum mechanics yields troublesome results. Problems arise when considering the collapse of the wave function, as in the Einstein-Podolsky-Rosen paradox.[4] Problems also arise when treating macroscopic systems, as in the Schrödinger cat paradox[5] and the Wigner paradox.[6] Finally, quantum theory is not overly compatible with general relativity.[7]

One way of imposing some quantum behavior on general relativity is commonly the following: The uncertainty relation for time and energy implies that at any point in space one can "borrow" any amount of energy from the vacuum if it is borrowed for a sufficiently short period of time. This energy fluctuation of the vacuum then gives rise to metric fluctuations via the general-relativity field equations. One then has, among other things, a vacuum filled with virtual Schwarzschild singularities. This can be awkward.

An alternative approach is to impose *ab initio* an uncertainty on the metric tensor, and to see if by that uncertainty the results of quantum mechanics can be deduced. As this paper will show, with a few not particularly unreasonable assumptions a large segment of the formalism of the quantum theory can be derived and, more importantly, understood.

Mathematical spaces with stochastic metrics have been investigated earlier by Schweizer[8] for Euclidean spaces, and by March[9,10] for Minkowski space. In a recent paper by Blokhintsev,[11] the effects on the physics of a space with a small stochastic component are considered. It is our goal, however, not to show the effects on physical laws of a stochastic space, but to show that the body of quantum mechanics can be deduced from simply imposing stochasticity on the space-time.

Our method will be to write down (in Sec. II) a number of statements (theorems, postulates, etc.) which are the fundamentals of the theory. We will then (in Sec. III) describe the statements and indicate proofs where the statements are theorems rather than postulates. Finally (in Sec. IV) we will derive some physical results, namely, the spread of the free particle (in empty space), the uncertainty principle, and the phenomenon of interference. The paper concludes (Sec. V) with a general discussion of the approach and a summary of results.

II. THE STATEMENTS

Statement 1, Mach's principle (*Frederick's version*).

1.1. In the absence of mass, space-time becomes not flat, but stochastic.

1.2. The stochasticity is manifested in a stochastic metric g_{ij}.

1.3. The mass distribution determines not only the space-time geometry, but also the space-time stochasticity.

1.4. The more mass in the space-time, the less stochastic the space-time becomes.

1.5. At the position of a mass "point", the space-time is not stochastic.

Statement 2, the contravariant observable theorem. All measurements of dynamical variables correspond to contravariant components of tensors.

Statement 3, the metric probability postulate. $P(x,t)=A\sqrt{-g}$, where for a one-particle system $P(x,t)$ is the particle probability distribution. A is a real-valued function, and g is the determinant of the metric.

Statement 4, the metric superposition postulate. If at the position of a particle the metric due to a specific physical situation is $g_{ik}(1)$ and the metric due to a different physical situation is $g_{ik}(2)$, then the metric at the position of the particle due to the presence of both of the physical situations is $g_{ik}(3)$,

$$g_{ik}(3)=\tfrac{1}{2}[g_{ik}(1)+g_{ik}(2)] .$$

Statement 5, the metric Ψ postulate. There exists a local complex diagonal coordinate system in which a component of the metric at the location of the particle in the wave function Ψ.

III. DESCRIPTION OF THE STATEMENTS

Statement 1, Mach's principle, is the basic postulate for our theory. It should only be added that requirement 1.5, that at the position of a mass point the space-time be not stochastic, is to ensure that an elementary mass particle (proton, quark, etc.) be bound.

Statement 2, the contravariant observable theorem, is also basic. It is contended, and the contention will be weakly proved, that measurements of dynamical variables are contravariant components of tensors. By this we mean that whenever a measurement can be reduced to a displacement in a coordinate system, it can be related to contravariant components of the coordinate system. Of course, if the metric g_{ik} is well known, one can calculate both covariant and contravariant quantities. In the real world, however, the quantum uncertainties in the mass distribution imply that the metric cannot be accurately known, so that measurements can only be reduced to contravariant quantities. Also, in our picture the metric is stochastic, so again we can only use contravariant quantities. We will verify the theorem for Minkowski space by considering an idealized measurement. Before we do, however, consider as an example the case of measuring the distance to a Schwarzschild singularity (a black hole) in the Galaxy. Let the astronomical distance to the object be $\bar{r}$ ($\equiv\bar{\xi}^1$). The covariant equivalent of the radial coordinate r is ξ_1, and

$$\xi_1=g_{1\nu}\xi^\nu=g_{11}\xi^1=\frac{r}{1-2Gm/r} ,$$

so that the contravariant distance to the object is

$$\text{distance}=\int_0^{\bar{r}} dr=\bar{r} ,$$

while the covariant distance is

$$\bar{\xi}_1=\int_0^{\bar{r}} d\left(\frac{r}{1-2Gm/r}\right)=\infty .$$

It is clear that here only the contravariant distance is observable.

Returning to the theorem, note that when one makes an observation of a dynamical variable (e.g. position, momentum, etc.), the measurement is usually in the form of a reading of a meter, or something similar. It is only through a series of calculations that one can reduce the datum to, say, a displacement in a coordinate system. For this reduction to actually represent a measurement (in the sense of Margenau[12]) it must satisfy two requirements. It must be instantaneously repeatable with the same results, and it must be a quantity which can be used in expressions to derive physical results (i.e., it must be a physically "useful" quantity). It will be shown that for Minkowski space, the derived "useful" quantity is contravariant.

Note first (Fig. 1) that for an oblique coordinate system, the contravariant coordinates of a point V are given by the parallelogram law of vector addition, while the covariant components are obtained by orthogonal projection onto the axes.[13]

We shall now consider an idealized measurement in special relativity, i.e., Minkowski space. Consider the space-time diagram of Fig. 2. We are given that in the coordinate system x', t', an object (m,n) is at rest. If one considers the situ-

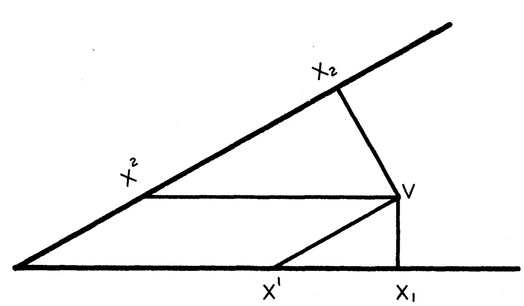

FIG. 1. Covariant and contravariant components in oblique coordinates.

ation from a coordinate system x,t traveling with velocity v along the x' axis, one has the usual Minkowski diagram[14] with coordinate axes Ox and Ot and velocity $v=\tan\alpha$ (where the units are chosen such that the velocity of light is unity). OC is part of the light cone.

Noting that the unprimed system is a suitable coordinate system in which to work, we now drop from consideration the original x',t' coordinate system.

We wish to determine the "length" of the object in the x,t coordinate system. Let it be arranged that at time $t(0)$ a photon shall be emitted from each end of the object (i.e., from points F and B). The emitted photons will intercept the t axis at times $t(1)$ and $t(2)$. The observer then deduces that the length of the object is $t(2)-t(1)$ (where $c=1$). The question is: What increment on the x axis is represented by the time interval $t(2)-t(1)$? One should note that the arrangement that the photons be emitted at time $t(0)$ is nontrivial, but that it can be done in principle. For the present, let it simply be assumed that there is a person on the object who knows special relativity and who knows how fast the object is moving with respect to the coordinate system. This person then calculates when to emit the photons so that they will be emitted simultaneously with respect to the x,t coordinate system.

Consider now Fig. 3, which is an analysis of the measurement. Figure 3 is just Fig. 2 with a few additions. The principal additions are the contravariant coordinates of F and B, x^1 and x^2 respectively. We assert, and it is easily shown, that $t(2)-t(1)=x^2-x^1$. This is seen by noticing that $x^2-x^1=$ line segment B,F, and that triangle $t(2)$, $t(0),Z$ is congruent to triangle $B,t(0),Z$. However, if we consider the covariant coordinates, we notice that $x_2-x_1=x^2-x^1$. This is not surprising since coordinate differences (such as x_2-x_1) are by definition (in flat space) contravariant quantities. To verify our hypothesis we must consider not coordinate differences which automatically satisfy the hypothesis, but the coordinates themselves. Consider a measurement not of the length of the object, but of the position (of the trailing edge m) of the object. Assume again that at time $t(0)$ a photon is emitted at F and is received at $t(1)$. The observer would then determine the position of m at $t(0)$ by simply measuring off the distance $t(1)-t(0)$ on the x axis. Notice that this would coincide with the contravariant quantity x^1. To determine the corresponding covariant quantity x_1, one would need to know the angle α (which is determined by the metric).

The metric g_{ik} is defined as $\hat{e}_i\cdot\hat{e}_k$, where $\hat{e}_i$ and $\hat{e}_k$ are the unit vectors in the directions of the coordinate axes x^i and x^k. Therefore, in order to consider an uncertain metric, we can simply consider that the angle α is unknown or uncertain. In this case measurement x^1 is still well defined $[x^1=t(1)-t(0)]$, but now there is no way to determine x_1 because it is a function of the angle α. In this case then, only the contravariant components of position are measureable. [It is also easy to see from geometry that if one were to use the covariant representation of $t(0)$, t_0, one could not obtain a metric-free-position measure of m.]

Statement 3, the metric probability postulate, can be justified by the following. Consider that there is given a sandy beach. Also given is one black grain of sand among the white grains of the beach. Further, if a number of observers on the beach had buckets of various sizes, and each of

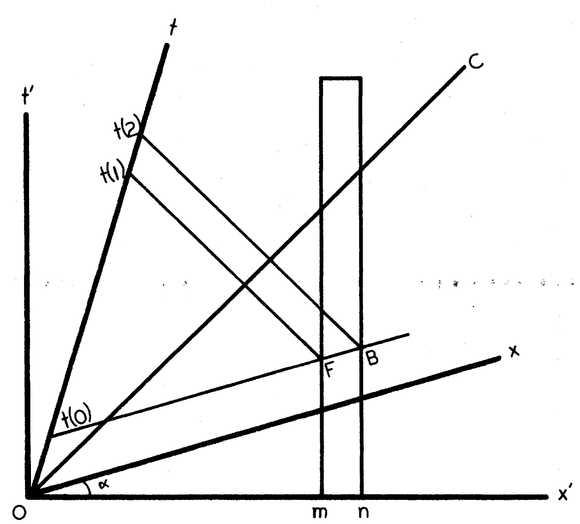

FIG. 2. An idealized measurement.

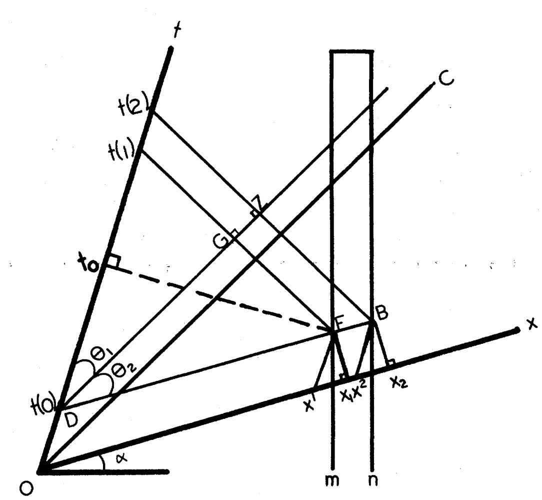

FIG. 3. Analysis of the idealized measurement.

the observers filled one bucket with sand, one could ask the following: What is the probability that a particular bucket contains the black grain? The probability would, of course, be proportional to the volume of the bucket.

Consider now the invariant volume element dV_I in Riemann geometry. One has that[15]

$$dV_I = \sqrt{-g}\, dx^1 dx^2 dx^3 dx^4 \ .$$

In other words, if a space with coordinates x^i had a fluctuating metric g_{jk}, the volume element dV_I would nonetheless be constant. It is reasonable, then, to take $\sqrt{-g}$ as the probability density for free space.

Consider again the sandy beach. Let the black grain of sand be placed on the beach by an aircraft dropping the grain over the geometric center of the beach. Now the location of the grain is not random. The probability of finding the grain increases as one proceeds toward the center of the beach, so that in addition to the volume of the bucket there is also a term in the probability function which depends on the distance to the beach center. In general, then, we expect the probability function $P(x,t)$ to be $P(x,t)=A\sqrt{-g}$, where A is a function analogous to the term in the example proportional to the distance from the center of the beach.

Statement 4, the metric superposition postulate, is adopted on the grounds of simplicity. Consider the metric (for a given set of coordinates) $g_{jk}^{s1}(x)$ due to a given physical situation $s1$ as a function of position x. Also let there be the metric $g_{jk}^{s1}(x)$ due to a different physical situation $s2$. What is the metric due to the simultaneous presence of situations $s1$ and $s2$? We are, of course, looking for a representation to correspond to quantum-mechanical linear superposition. The obvious and most simple assumption is that

$$g_{jk}^{s1} = \tfrac{1}{2}\left[g_{jk}^{s1}(x) + g_{jk}^{s2}(x)\right] \ .$$

However, this assumption is in contradiction with general relativity, a theory which is nonlinear in g_{jk}. The linearized general theory is still applicable. Therefore, the metric superposition postulate is to be considered as an approximation to an as yet unspecified full theory, valid over small distances in empty or almost empty space. Therefore, we expect that the quantum-mechanical principle of linear superposition will break down at some range. (This may eventually be the solution to the linear-superposition-type paradoxes in quantum mechanics.)

Statement 5, the metric Ψ postulate, is not basic to the theory. It exists simply as an expression of the following: There are at present two separate concepts, the metric g_{jk} and the wave function Ψ. It is an aim of this geometrical approach to be able to express one of these quantities in terms of the other. The statement that in some arbitrary complex coordinate transformation the wave function is a component of the metric is just a statement of this aim.

IV. PHYSICAL RESULTS

We derive first the motion of a test particle in an otherwise empty space-time. The requirement that the space is empty implies that the points in this space are indistinguishable. Also, we expect that, on the average, the space (since it is mass-free) is Minkowski space.

Consider the metric tensor at a point Θ_1. Let the metric at Θ_1 be $\tilde{g}_{\mu\nu}$ (a tilde over a quantity indicates that it is stochastic). Since $\tilde{g}_{\mu\nu}$ is stochastic, the metric components do not have well-defined values. We cannot then know $\tilde{g}_{\mu\nu}$, but we can ask for $P(g_{\mu\nu})$, which is the probability of a particular metric $\tilde{g}_{\mu\nu}$. Note then that for the case of empty space, $P_{\Theta 1}(g_{\mu\nu}) = P_{\Theta 2}(g_{\mu\nu})$, where $P_{\Theta 1}(g_{\mu\nu})$ is to be interpreted as the probability of metric $g_{\mu\nu}$ at point $\Theta 1$.

If one inserts a point test particle into the space-time, with a definite position and (ignoring quantum mechanics for a moment) momentum, the particle motion is given by the Euler-Lagrange equations,

$$\ddot{x}^i + \left\{ {i \atop jk} \right\} \dot{x}^j \dot{x}^k = 0,$$

where $\left\{ {i \atop jk} \right\}$ are the Christoffel symbols of the second kind, and where $\dot{x}^j \equiv dx^j/ds$ and s can be either proper time or any single geodesic parameter.[17] Since $\tilde{g}_{\mu\nu}$ is stochastic, these equations generate not a path but an infinite collection of paths, each with a distinct probability of occurrence from $P(g_{\mu\nu})$. (That is to say $\left\{ {i \atop jk} \right\}$ is stochastic; $\left\{ {\tilde{i} \atop jk} \right\}$.)

In the absence of mass, the test-particle motion is easily soluble. Let the particle initially be at (space) point Θ_0. After time dt, the Euler-Lagrange equations yield some distribution of position $D_1(x)$. [$D_1(x)$ represents the probability of the particle being in the region bounded by x and $x+dx$.] After another interval dt, the resulting distribution is $D_{1+2}(x)$. From probability theory,[18] this is the convolution,

$$D_{1+2}(x) = \int_{-\infty}^{\infty} D_1(y) D_2(x-y)\, dy \ ,$$

but in this case, $D_1(x) = D_2(x)$. This is so because the Euler-Lagrange equations will give the same distribution $D_1(x)$ independently of at which point one propagates the solution. This is, of course, because

$$g_{\mu\nu}(x) \equiv \{ g_{\mu\nu}(x_1), g_{\mu\nu}(x_2), \ldots \}$$

are identically distributed random variables. Thus,

$$D_i(x) \equiv \{D_1(x), D_2(x), \ldots\}$$

are also identically distributed random variables. The motion of the test particle is the repeated convolution $D_{1+2+\cdots}(x)$, which by the central-limit theorem[18] is a normal distribution. Thus the position spread of the test particle at any time $t > 0$ is a Gaussian. The spreading velocity is found as follows. After N convolutions (N large), one obtains a normal distribution with variance σ^2, which again by the central-limit theorem is N times the variance of $D_1(x)$. Call the variance of $D_1(x)$ a,

$$\mathrm{Var}(D_1) = a \; .$$

The distribution D_1 is obtained after time dt. After N convolutions,

$$\Delta x = \mathrm{Var}\left(D_{\sum_i^n i}\right) = Na \; .$$

This is obtained after N time intervals dt. One then has

$$\frac{\Delta x}{\Delta t} = \frac{Na}{N} \; ,$$

which is to say that the initially localized test particle spreads with a constant velocity a. In order that the result be frame independent, $a = c$, and one then has the results of quantum mechanics. At the beginning of this derivation it was given that the particle had an initial well-defined position and also momentum. If for the benefit of quantum mechanics we had specified a particle with a definite position, but with a momentum distribution, one would have obtained the same result but with the difference of having a different distribution D_1 due to the uncertainty of the direction of propagation of the particle.

In the preceding, we have made use of various equations. It is then appropriate to say a few words about what equations mean in a stochastic space-time.

Since in our model the actual points of the space-time are of a stochastic nature, these points cannot be used as a basis for a coordinate system, nor, for that reason, can derivatives be formed. However, the space-time of common experience (i.e., the laboratory frame) is nonstochastic in the large. It is only in the micro world that the stochasticity is manifested. One can then take this large-scale nonstochastic space-time and mathematically continue it into the micro region. This mathematical construct provides a nonstochastic space to which the stochastic physical space can be referred.

The (physical) stochastic coordinates $\tilde{x}^i$ then are stochastic only in that the equations transforming from the laboratory coordinates x^1 to the physical coordinates $\tilde{x}^1$ are stochastic.

For the derivation of the motion of a free particle we used Statement 1, Mach's principle. We will now use also Statement 2, the contravariant observable theorem, and derive the uncertainty principle for position and momentum. Similar arguments can be used to derive the uncertainty principle for other pairs of conjugate variables. It will also now be shown that there is an isomorphism between a variable and its conjugate, and covariant and contravariant tensors.

We assume now that we are able to define a Lagrangian L. One defines a pair of conjugate variables as usual,

$$P_j = \frac{\partial L}{\partial \dot{q}^j} \; .$$

Note that this defines P_j, a covariant quantity, so that a pair of conjugate variables so defined contains one covariant and one contravariant member (e.g. P_j and q^j). But since P_j is covariant, it is not observable in the laboratory frame. The observable quantity is just

$$\tilde{P}^j = \tilde{g}^{j\nu} P_\nu \; ,$$

but $\tilde{g}_{\mu\nu}$ is stochastic so that $\tilde{P}^j$ is a distribution. Thus if one member of an observable-conjugate-variable pair is well defined, the other member is stochastic. By observable conjugate variables we mean not, say, P_j, q^j derived from the Lagrangian, but the observable quantities $\tilde{P}^j, q^j$, where $\tilde{P}^j = \tilde{g}^{j\nu} P_\nu$; i.e. both members of the pair must be contravariant.

However, we can say more. Indeed, we can derive an uncertainty relation. Consider

$$\Delta q^1 \Delta P^1 = \Delta q^1 \Delta (p_\nu \tilde{g}^{\nu 1}).$$

What is the minimum value of this product, given an initial uncertainty Δq^1? Since P_j is an independent variable, we may take $\Delta P_j = 0$ so that

$$\Delta P^1 = \Delta (P_\nu \tilde{g}^{\nu 1}) = P_\nu \Delta \tilde{g}^{\nu 1} \; .$$

In order to determine $\Delta \tilde{g}^{\nu 1}$ we will argue that the variance of the distribution of the average of the metric over a region of space is inversely proportional to the volume,

$$\mathrm{Var}\left(\frac{1}{V}\int_V \tilde{g}_{\mu\nu} dv\right) = \frac{k}{V} \; .$$

In other words, we wish to show that if we are given a volume and if we consider the average values of the metric components over this volume, then these average values, which of course are stochastic, are less stochastic than the metric component values at any given point in the volume. Further, we wish to show that the stochasticity, which we can represent by the variances of the distributions of the metric components, is inversely proportional to the volume. This allows that over macroscopic volumes the metric behaves classically (i.e., according to general relativity).

For simplicity, let the distribution of each metric component at any point Θ be normal,

$$f_{\tilde{g}_{\mu\nu}}(g_{\mu\nu})=\frac{1}{\sigma(2\pi)^{1/2}}\exp\left[-\frac{1}{2}\left(\frac{g_{\nu\mu}}{\sigma}\right)^2\right] .$$

Note also that if $f(y)$ is normal the scale transformation $y\rightarrow y/m$ results in $f(y/m)$ which is normal with

$$\sigma^2_{(y/m)}=\frac{\sigma^2_y}{m^2} .$$

Also, for convenience, let

$$f_{g_{\mu\nu}\text{ at }\Theta_1}(g_{\mu\nu})\equiv f_{\Theta_1}(g_{\mu\nu}) .$$

We now require

$$\operatorname{Var} f_{((\Theta_1+\Theta_2+\cdots+\Theta_m)/m)}\equiv\sigma^2_{((\Theta_1+\Theta_2+\cdots+\Theta_m)/m)} ,$$

where $f_{(\Theta_1)}$ is normally distributed. Now again the convolute $f_{(\Theta_1+\Theta_2)}(g_{\mu\nu})$ is the distribution of the sum of $g_{\mu\nu}$ at Θ_1 and $g_{\mu\nu}$ at Θ_2,

$$f_{(\Theta_1+\Theta_2)}=\int_{-\infty}^{\infty}f_{\Theta_1}(g^1_{\mu\nu})f_{\Theta_2}(g^1_{\mu\nu}-g^2_{\mu\nu})dg^2_{\mu\nu} ,$$

where $g^1_{\mu\nu}$ is defined to be $g_{\mu\nu}$ at Θ_1. Here, of course, $f_{\Theta_1}=f_{\Theta_2}$ as the space is empty so that

$$f_{(\Theta_1+\Theta_2)/2}=f_{(g_{\mu\nu}/m\text{ at }\Theta_1+g_{\mu\nu}/2\text{ at }\Theta_2)}$$

is the distribution of the average of $g_{\mu\nu}$ at Θ_1 and $g_{\mu\nu}$ at Θ_2. $\sigma^2_{(\Theta_1+\Theta_2+\cdots+\Theta_m)}$ is easily shown from the theory of normal distributions to be

$$\sigma^2_{(\Theta_1+\Theta_2+\cdots+\Theta_m)}=m\sigma_\Theta .$$

Also, $f_{(\Theta_1+\Theta_2+\cdots+\Theta_m)}$ is normal. Hence,

$$\sigma^2_{((\Theta_1+\Theta_2+\cdots+\Theta_m)/m)}=\frac{m\sigma^2_\Theta}{m^2}=\frac{\sigma^2_\Theta}{m} ,$$

or the variance is inversely proportional to the number of elements in the average, which in our case is proportional to the volume. For the case where the distribution $f_{(g_{\mu\nu})}$ is not normal, but also not "pathological", the central-limit theorem gives the same results as those obtained for the case where $f_{(g_{\mu\nu})}$ is normal. Further, if the function $f_{(g_{\mu\nu})}$ is not normal, the distribution $f_{((\Theta_1+\Theta_2+\cdots+\Theta_m)/m)}$ in the limit of large m is normal,

$$f_{((\Theta_1+\Theta_2+\cdots+\Theta_m)/m)}\rightarrow f_{((\int_V\tilde{g}_{\mu\nu}dv)/V)} .$$

In other words, over any finite (i.e., noninfinitesimal) region of space, the distribution of the average of the metric over the region is normal. Therefore, in as far as we do not consider particles to be "point" sources, we may take the metric fluctuations at the location of a particle as normally distributed for each of the metric components $\tilde{g}_{\mu\nu}$. However, note that this does not imply that the distributions for any of the metric tensor components are the same for there is no restriction on the value of the variance σ^2 (e.g., in general, $f_{(\tilde{g}_{11})}\neq f_{(\tilde{g}_{22})}$). Note also that the condition of normally distributed metric components does not restrict the possible particle probability distributions, save that they be single-valued and non-negative. This is equivalent to the easily proved statement that the functions

$$f_{(x,\alpha,\sigma)}=\frac{1}{(2\pi\sigma)^{1/2}}\exp\left[-\frac{1}{2}\left(\frac{x-\alpha}{\sigma}\right)^2\right]$$

are complete for non-negative functions.

Having established that

$$\operatorname{Var}\left(\frac{\Theta_1+\Theta_2+\cdots+\Theta_m}{m}\right)=\frac{\sigma^2_\Theta}{m} ,$$

consider again the uncertainty product,

$$\Delta q^1\Delta P^1=P_\nu\Delta q^1\Delta\tilde{g}^{\nu 1} .$$

Δq^1 goes as the volume [volume here is V^1 (the one-dimensional volume]. $\Delta\tilde{g}^{\nu 1}$ goes inversely as the volume so that $P_\nu\Delta q^1\Delta\tilde{g}^{\nu 1}$ is independent of the volume; i.e. as one takes q^1 to be more localized, P^1 becomes less localized by the same amount, so that for a given covariant momentum P_j (which we will call the proper momentum), $P_\nu\Delta q^1\Delta g^{\nu 1}=k$, if also P_ν is uncertain,

$$P_\nu\Delta q^1\Delta g^{\nu 1}\geq k .$$

The fact that we have earlier shown that a free particle spreads indicates the presence of a minimum proper momentum. If the covariant momentum were zero, then the observable contravariant momentum $P^\nu=g^{\nu\mu}P_\mu$ would also be zero and the particle would not spread. Hence,

$$P_{\nu\min}\Delta q^1\Delta g^{1\nu}=k_{\min} ,$$

or in general,

$$\Delta q^1\Delta(P_{\nu\min}g^{\nu 1})=\Delta q^1\Delta P^1>k_{\min} ,$$

which is the uncertainty principle.

With the usual methods of quantum mechanics, one treats as fundamental not the probability density $P(x,t)$, but the wave function Ψ [$\Psi^*\Psi=P(x,t)$ for the Schrödinger equation]. The utility of using Ψ is in that Ψ contains phase information. Hence, by using Ψ the phenomenon of interference is possible. It might be thought that our stochastic space-time approach, as it works directly with $P(x,t)$, might have considerable difficulty in producing interference. In the following, it will be shown that Statements 3 and 4 can produce interference in a particularly simple way.

Consider again the free particle in empty space. By considering the metric only at the location of the particle, we can suppress the stochasticity by means of Statement 1.5. Let the metric at the location of the particle be $g_{\mu\nu}$. We assume, at present, no localization, so that the probability distribution $P(x,t)=$ constant. $P(x,t)=A\sqrt{-g}$ by Statement 3. Here A is just a normalization constant so that $\sqrt{-g}=$ constant. We can take the constant equal to unity.

Once again, the condition of empty space implies that the average value of the metric over a region of space-time approaches the Minkowski metric as the volume of the region increases.

Now consider, for example, a two-slit experiment in this space-time. Let the situation where only slit one is open result in a metric $g^{s1}_{\mu\nu}$. Let the situation where only slit two is open result in a metric $g^{s2}_{\mu\nu}$. The condition that both slits are open then, by Statement 4, results in

$$g^{s3}_{\mu\nu}=\tfrac{1}{2}(g^{s1}_{\mu\nu}+g^{s2}_{\mu\nu}) .$$

Let us also assume that the screen in the experiment is placed far from the slits so that the individual probabilities $(-g^{s1})^{1/2}$ and $(-g^{s2})^{1/2}$ can be taken as constant over the screen.

Finally, let us assume that the presence of the two-slit experiment in the space-time does not appreciably alter the situation that the metrics $g^{s1}_{\mu\nu}$ and $g^{s2}_{\mu\nu}$ are in the average $\eta_{\mu\nu}$ that is to say that the insertion of the two-slit experiment does not appreciably change the geometry of space-time).

It is of interest to ask what one can say about the metric $g^{s1}_{\mu\nu}$. If the particle is propagated in, say, the x^3 direction and, of course, also in the x^4 direction, we might expect that the metric be equal to the Minkowski metric $\eta_{\mu\nu}$, save for g_{33} and g_{44}. We will then take the following:

$$\tilde{g}^{s1}_{\mu\nu}=\left\|\begin{matrix}1&0&0&0\\0&1&0&0\\0&0&s&0\\0&0&0&-t\end{matrix}\right\| ,$$

and

$$|\tilde{g}^{s1}_{\mu\nu}|=-st ,$$

where s and t are as yet unspecified functions of position. In order that $|\tilde{g}^{s1}_{\mu\nu}|$ be constant, let $s=t^{-1}$ so that

$$|g^{s1}_{\mu\nu}|=|\eta_{\mu\nu}|=-1 .$$

Now we will introduce an unphysical situation, the utility of which will be seen shortly.

Let $s=e^{i\alpha}$, where α is some unspecified function of position. Consider the following metrics:

$$\tilde{g}^{s1}_{\mu\nu}=\left\|\begin{matrix}1&0&0&0\\0&1&0&0\\0&0&e^{i\alpha}&0\\0&0&0&-e^{-i\alpha}\end{matrix}\right\| ,$$

$$\tilde{g}^{s2}_{\mu\nu}=\left\|\begin{matrix}1&0&0&0\\0&1&0&0\\0&0&e^{i\beta}&0\\0&0&0&-e^{-i\beta}\end{matrix}\right\| ,$$

where β is again some unspecified function of position;

$$(-|\tilde{g}^{s1}_{\mu\nu}|)^{1/2}=(-|\tilde{g}^{s2}_{\mu\nu}|)^{1/2}=1 \qquad (\text{note } |\tfrac{1}{2}A_{jk}|=\tfrac{1}{16}|A_{jk}|),$$

$$(-|\tilde{g}^{s3}_{\mu\nu}|)^{1/2}=(\tfrac{-1}{16}|\tilde{g}^{s1}_{\mu\nu}+\tilde{g}^{s2}_{\mu\nu}|)^{1/2}=[\tfrac{-1}{16}(2+e^{i(\alpha-\beta)}+e^{-i(\alpha-\beta)})]^{1/2},$$

$$(-|\tilde{g}^{s3}_{\mu\nu}|)^{1/2}=\tfrac{1}{2}|\cos\tfrac{1}{2}(\alpha-\beta)| ,$$

where $|\,|$ indicates absolute value. This is, of course, the phenomenon of interference. The metrics $\tilde{g}^{s1}_{\mu\nu}$, $\tilde{g}^{s2}_{\mu\nu}$ and $\tilde{g}^{s3}_{\mu\nu}$ describe, for example, the two-slit experiment described previously. The analogy of the functions $e^{i\alpha}$ and $e^{-i\alpha}$ with Ψ and Ψ^* is obvious. The use of complex functions in the metric, however, is unphysical. The resultant line element $ds^2=\tilde{g}_{\mu\nu}dx^\mu dx^\nu$ would be complex and hence unphysical. The following question arises: Can we reproduce the previous arguments, but with real functions? The answer is yes, but first we must digress briefly to discuss quadratic-form matrix transformations.[17]

Let

$$X=\left\|\begin{matrix}dx^1\\dx^2\\dx^3\\dx^4\end{matrix}\right\|$$

and again let

$$G=\|g_{\mu\nu}\| .$$

Then $X^tG\,X=ds^2=g_{\mu\nu}dx^\mu dx^\nu$, where X^t is the transpose of X. Consider transformations which leave the line element ds^2 invariant. Given a transformation matrix W,

$$X=WX' ,$$

and

$$X^tGX=X'^tG'X'=(X^t(W^t)^{-1})G'(W^{-1}X)$$

[note: $(WX')^t = X'^t W'^t$]. However, $X^t G X \equiv (X^t (W^t)^{-1})(W^t G W)(W^{-1} X)$ so that

$$G' = W^t G W .$$

In other words, the transformation W takes G into $W^t G W$. Now in the transformed coordinates a metric $\tilde{g}^{s1}_{\mu\nu} \equiv G^{s1}$ goes to $W^t G^{s1} W$. Therefore,

$$\Psi_1^* \Psi_1 = (-|W^t G^{s1} W|)^{1/2}$$
$$= (-|W^t|\,|g^{s1}_{\mu\nu}|\,|W|)^{1/2} ,$$
$$\Psi_3^* \Psi_3 = (-\tfrac{1}{16}|W^t G^{s1} W + W^t G^{s2} W|)^{1/2}$$
$$= [-\tfrac{1}{16}|W^t (G^{s1} + G^{s2}) W|]^{1/2}$$
$$= (-\tfrac{1}{16}|W^t|\,|G^{s1} + G^{s2}|\,|W|)^{1/2} .$$

If we can find a transformation matrix W with the properties

(i) $|W| = 1$,
(ii) W is not a function of α or β,
(iii) $W^t G W$ is a matrix with only real components,

then we will again have the interference phenomenon with $g'_{\mu\nu}$ real, and again $\Psi_1^*\Psi_1 = \Psi_2^*\Psi_2 = 1$ and

$$\Psi_3^* \Psi_3 = \tfrac{1}{2}\left|\cos\left(\frac{\alpha - \beta}{2}\right)\right| .$$

The appropriate matrix W is

$$\left\|\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{matrix}\right\| = W .$$

If, as previously,

$$\left\|\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\alpha} & 0 \\ 0 & 0 & 0 & -e^{-i\alpha} \end{matrix}\right\| = \|\tilde{g}^{s1}_{\mu\nu}\| ,$$

then

$$\left\|\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\cos\alpha & \sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{matrix}\right\| = \|\tilde{g}'^{s1}_{\mu\nu}\| = W^t \tilde{g}^{s1}_{\mu\nu} W ,$$

so that in order to reproduce the phenomenon of interference, the stochastic metric $\tilde{g}_{\mu\nu}$ will have off-diagonal terms. Incidentally, the coordinates appropriate to $\tilde{g}^{s1}_{\mu\nu}$ are

$$x^{1\prime} = x^1 ,$$
$$x^{2\prime} = x^2 ,$$
$$x^{3\prime} = \frac{-i}{\sqrt{2}} x^3 + \frac{1}{\sqrt{2}} x^4 ,$$
$$x^{4\prime} = \frac{1}{\sqrt{2}} x^3 - \frac{i}{\sqrt{2}} x^4 ,$$

which is to say that with an appropriate coordinate system which is complex, we can treat the free-space probability distributions $\Psi^*\Psi$ in a particularly simple way. Since the components x^μ do not appear in predictions (such as $\Psi^*\Psi$), we may simply, as an operational convenience, take $\tilde{g}_{\mu\nu}$ to be diagonal, but with complex components.

V. DISCUSSION

Having recognized that quantum mechanics is merely an operational calculus, and also having observed that general relativity is a true theory of nature with both an operational calculus and a *Weltanschauung*, it has been attempted to generate quantum mechanics from considerations of the structure of space-time. As a starting point we have used a version of Mach's principle where in the absence of mass space-time does not become flat, but becomes undefined (or more exactly, not well defined) such that $P_\Theta(g_{\mu\nu})$ is at a given point Θ the probability [in the Copenhagen sense[18]] distribution for $g_{\mu\nu}$. From this, the motion of a free (test) particle was derived. This is a global approach to quantum theory. It should be noted that there are two logically distinct approaches to conventional quantum mechanics: a local formalism and a global formulation. The local formalism relies on the existence of a differential equation (such as the Schrödinger equation) describing the physical situation (e.g. the wave function of the particle) at each point in space-time. The existence of this equation is operationally very convenient. On the other hand, the global formulation (or path formulation, if you will) is rather like the Feynman path formalism for quantum mechanics,[19] which requires the enumeration of the "action" over these paths. This formulation is logically very simple, but operationally it is exceptionally complex. What is required is a local formalism. Statement 3, $P(x,t) = A\sqrt{-g}$, is local and provides the basis for the further development of stochastic space-time quantum theory. Statements 1 and 3 are then logically related. The remaining Statements 2, 4, and 5 are secondary in importance.

The conclusion is that with the acceptance of the statements, the following can be deduced:

(i) the motion of a free particle, and the spread of the wave packet,
(ii) the uncertainty principle,
(iii) the nature of conjugate variables,
(iv) interference phenomena,

(v) an indication of where conventional quantum mechanics breaks down (i.e. the limited validity of linear superposition).

This paper represents an early stage of a theory. What is ultimately required is a set of "field" equations (analogous to the general-relativity field equations) which relate the mass distribution in the space-time to the stochasticity so that one can calculate $P(x,t)$ for all situations.

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